blast-wave theory[3] with a single modification. The essential details are as follows.

The total energy of the flow field, that is, the energy of the region bounded by the outer shock on one side and the orifice plate on the other, may be written

$$E = \int_{-(\pi/2)}^{\pi/2}\int_0^{R_s}\left[e + \frac{1}{2}(u^2 + v^2)\right]\rho r dr d\theta \qquad (1)$$

where e is the specific internal energy, u and v are the velocity components, and ρ is the density.

The similarity assumption is now introduced by the following substitutions:

$$\eta = \frac{r}{R_s} \qquad u = \dot{R}_s\bar{u} \qquad v = \dot{R}_s\bar{v}$$
$$e = \dot{R}_s^2\bar{e} \qquad \rho = \rho_0\bar{\rho}$$

Then,

$$E = R_s^2\dot{R}_s^2\rho_0\int_{-(\pi/2)}^{\pi/2}\int_0^1\left[\bar{e} + \frac{1}{2}(\bar{u}^2 + \bar{v}^2)\right]\bar{\rho}\eta d\eta d\theta \quad (2)$$

It can be shown that explicit time dependance of the equations of motion can be eliminated if

$$R_s \sim t^N \qquad (3)$$

where t is time. Also, since the orifice is choked,

$$E = \zeta h t \qquad (4)$$

where ζ is the energy flux through the orifice and h is the orifice height. Equation (4) is the point of departure from the usual blast-wave theory in which E is supposed constant. The present approach is analogous to the steady hypersonic flow over a slender body whose drag is proportional to its length.[3]

Equations (2–4) lead to

$$R_s \sim (\zeta h/\rho_0)^{1/4}\, t^{3/4} \qquad (5)$$

It is implicit in the derivation that the integrand of Eq. (2) is a universal constant. However, the previous discussion of the schlieren photographs leads to the conclusion that, although the integrand is independent of time, it does depend on the operating conditions. Thus, only the time dependence predicted by Eq. (5) can be expected to be correct. The value of N for the constant energy case is $\frac{1}{2}$; if total momentum flux, rather than energy flux, is conserved, an answer of $\frac{2}{3}$ is obtained.

In the light of the previous predictions, the experimental results can now be considered. Figure 4 shows the shock radius in inches plotted against time in microseconds for two sets of conditions. The initial pressure in the shock tube is P_1, and therefore, since there was no orifice diaphragm, it is the pressure into which the gas from the orifice expands. The Mach number of the driving shock wave is M_1. For each set of points, two lines have been drawn matching the data at arbitrary points. The solid line corresponds to energy conservation and therefore has a slope of $\frac{3}{4}$, whereas the broken line corresponds to momentum conservation and has a slope of $\frac{2}{3}$. Clearly, the experimental points correspond closely to these values of N. The scatter and experimental uncertainty preclude the favoring of either of these values, but it should be pointed out that the conservation of energy in blast-wave problems is favored by both experiment and more refined theories.[4,5] Two other lines are shown on Fig. 4 for comparative purposes. One represents the constant energy solution, and the other represents the acoustic limit.

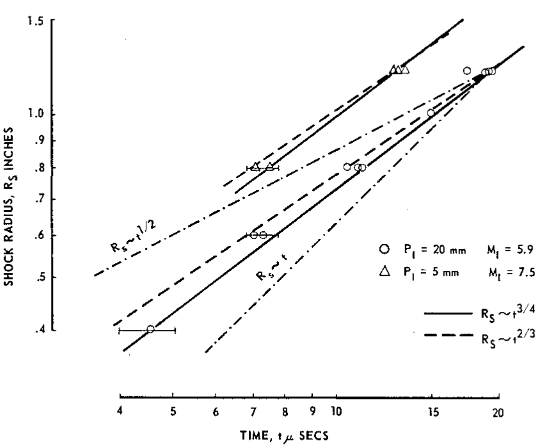

Fig. 4 Shock radius vs time history.

The calculated scaling factor for the two sets of data is approximately 1.1 in the shock radius and is clearly far too small. This lack of agreement is consistent with the earlier observations.

The evidence presented so far strongly favors the existence of flow-field similarity. However, within the nonsteady flow, a steady flow region must develop. At very early times, this will be bounded by a Mach wave of velocity $(u - a)$. After a few microseconds (much less than the 10–20 μsec flow time associated with Fig. 2), this wave will cross the inner shock, and this shock wave will then become the boundary between the steady and nonsteady flow. Now a steady flow cannot be matched with a self-similar flow across this shock wave. Consequently, a satisfactory explanation of the observed scaling is yet to be forthcoming, and further experimental work seems desirable. It should be noted that this point does not throw doubt on the blast-wave analysis, since the major portion of the energy resides close to the outer shock, between the shock and the interface.

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Equilbrium Turbulent Boundary Layers

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For the past few years, the writer and D. M. Gibson have been engaged in the study of equilibrium turbulent boundary layers. An early version of the work appeared in 1962[1] but the final work is only recently available.[2,3] Since the work has been so long in evolving and since it may not be available in the open literature for a while yet, a short summary will be provided here. This summary has been prompted particularly by the recent work of Libby, Baronti, and Napolitano[4] where, indeed, reasonable predictions of data have been accomplished. However, in comparison, the writer believes his own work to be analytically more precise, physically simpler, and amendable to further extensions. Moreover, unlike Refs. 1 and 4, Refs. 2 and 3 include cases of

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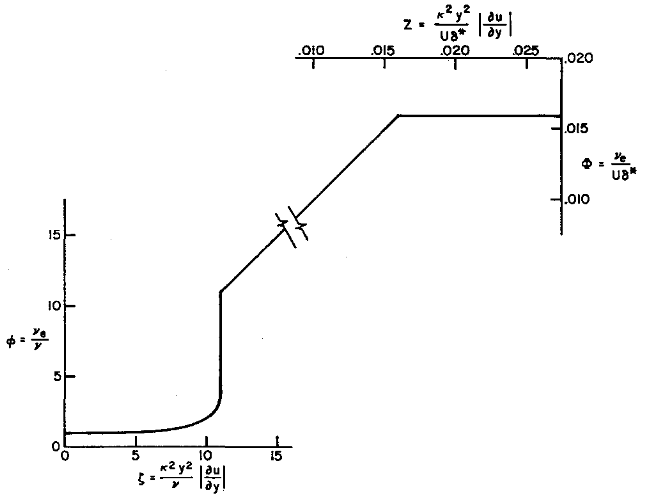

Fig. 1 The complete effective viscosity function. Note that $\varphi(\zeta) = R\Phi(Z)$, $\zeta = RZ$ where $R = U\delta^*/\nu$. The overlap portion increases with R.

small wall shear stress τ_0 and incipient separation. Although not stated explicitly, the analysis of Refs. 1 and 4 is limited to values of $\nu(dp/dx)/\rho\, u_\tau^3$ less than about 0.05; physically this corresponds to the disappearance of the logarithmic portion of the law of wall.

The basis of the work reported in Refs. 2 and 3 is a completely defined hypothesis for the effective or eddy viscosity ν_e. Thus,

$$\nu_e/\nu = \varphi[(\kappa^2 y^2/\nu)(\partial u/\partial y)] \qquad \text{for small } y \qquad (1a)$$

$$\nu_e/U\delta^* = \Phi[(\kappa^2 y^2/U\delta^*)(\partial u/\partial y)] \qquad \text{for large } y \qquad (1b)$$

If an overlap region exists where both functions apply, then it follows that $\nu_e = \nu\varphi = U\delta^*\Phi = \kappa^2 y^2 \partial u/dy$. This, of course, is a result identical to that given by Prandtl's mixing length theory. Furthermore, $\varphi \rightarrow 1$ as $y \rightarrow 0$, and the complete function φ may be obtained with data at zero pressure gradient. For large enough y, $\Phi = 0.016$ in accordance with Clauser's hypothesis for the outer layer. Figure 1 summarizes the complete hypothesis, the basis of which is considered in more detail in Ref. 3.

The complete equations of motion have been solved numerically. A slight, a posteriori, approximation† is made so that the final results appear directly in the following simple forms: for the defect layer,

$$(U - u)/u_\tau = f'[(y/\Delta), \beta] \qquad (2a)$$

and for the wall layer,

$$u/u_\tau = u^+ [(u_\tau y/\nu), \alpha] \qquad (2b)$$

where

$$\Delta \equiv \int_0^\infty \left[\frac{(U-u)}{u_\tau}\right] dy, \qquad \beta \equiv \frac{\delta^*}{\tau_0}\frac{dp}{dx}$$

in accordance with Clauser's usage; $u_\tau \equiv (\tau_0/\rho)^{1/2}$ and $\alpha = \nu(dp/dx)/\rho\, u_\tau^3$.

It should be noted that the functions f' and u^+ automatically overlap for a significant portion of the boundary layer and not at a single point. The overlap portion is not necessarily logarithmic.

In the overlap region, we find that $U/u_\tau = (2/c_f)^{1/2} = f' + u^+ = fcn(\beta, U\delta^*/\nu)$ so that a comparatively simple skin-friction equation is directly obtained.

All of the preceding equations may be presented in a form that remains finite as $\beta \rightarrow \infty$ or $u_\tau \rightarrow 0$.

† We actually find that $(U - u)/u_\tau = f'(y/\Delta, \beta, u_\tau/U)$, but the slight dependency on u_τ/U may be eliminated within a known and negligible error.

Although analytically more precise, and without benefit of empirical adjustment, as β (or the parameter S of Ref. 4) is varied, agreement with data is better than that obtained in Ref. 4. Furthermore, it was possible to predict the data of Stratford for $1/\beta = 1/\alpha = 0$.

Equilibrium turbulent boundary layers are described simply and directly by the condition $\beta(x) \equiv \delta^*(dp/dx)/\tau_0 = \text{const}$ and without recourse to streamwise integrations of the momentum integral equation. Furthermore, it is determined that equilibrium turbulent boundary layers exist in the range $-0.5 \leq \beta \leq \infty$ (the lower limit actually varies slightly with $U\delta^*/\nu$).

It is believed that the work described in Refs. 2 and 3 is a precursor to a general turbulent boundary-layer theory that correctly identifies separation as the point where $\tau_0(x) = 0$. Further extensions to heat and mass transfer are readily foreseeable and will be pursued.

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Nonequilibrium Sodium Ionization in Laminar Boundary Layers

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Introduction

An ablation heat shield in its manufacture can be contaminated with very small amounts of highly ionizable compounds, which, although insignificant with respect to their total weight, can become dominant in the production of electrons. These trace contaminants can then determine the observable characteristics of a re-entry vehicle. It is the purpose of this note to describe a method for determining the boundary-layer ionization due to trace contaminants.

Basic Equation and Solution

Consider a contaminant being injected into the boundary layer along a flat plate in small enough amounts so that it does not significantly alter the velocity and enthalpy profiles that will be determined by the gross amount of blowing. If the ions, atoms, and electrons associated with the trace contaminant are treated as one species, its distribution through the boundary layer can be represented by the following similar profile:

$$S = \int_0^\eta [f''(\eta)]^{Sc} d\eta \Big/ \int_0^\infty [f''(\eta)]^{Sc} d\eta \qquad (1)$$

where

$$S = (C_{Aw} - C_A)/C_{Aw} \qquad (2)$$

and where Sc is the Schmidt number. The governing species

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